Game Theory 9

SLOB Mapped against the Module

To equip oneself with application-oriented knowledge of Game Theory to facilitate management decisions for optimisation through resource allocation, managing competition, work scheduling and managing cost overrun, demand estimation, production and cost analysis etc.

Module Learning Objectives

After studying this module, the students will be able to:

- Understand situations where decisions are to be made under conditions of conflict.
- Understand the different procedures used in the selection and execution of various strategies which result in winning the Game.

Game Theory

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In decision making often we come across situations where two or more opposing parties are seen to have conflicting interests. Action of one depends on the action taken by the opponent. Any Military Operation is an example of such a conflicting situation. Each participant of the operation takes all possible measures to prevent the opponent from succeeding. Situations in the field of Economics, particularly when there is free competition belong to the class of conflicts, too. Firms trying to maintain their market share, work in a conflicting or competitive environment and any move by one is suitably counter moved by the other. To increase the market share if one firm takes a strategy of reducing the selling price of its product by giving some discount then the other might take a strategy to redesign its product and increase its value at a price lower than the competitor.

To analyse such conflicting situations, some special mathematical model called "Game Theory" is used. It was first developed, to solve problems in economics, by Hungarian born American mathematician John von Neumann and his Princeton University colleague Oskar Morgenstern, a German born American economist in the year 1944. They observed that economics is much like a game, wherein players anticipate each other's moves. They named it Game Theory which is somewhat of a misnomer because it does not share the fun or frivolity associated with games.

Game Theory may be defined as a type of Decision Making situation when two or more intelligent and rational opponents are involved under conditions of conflict and competition. It is a type of Decision Theory in which one's choice of action is determined after taking into account all possible alternatives available to the opponent participating in the same game. Game Theory does not insist on how a game should be played but tells the procedure and principles by which action should be selected. 'Game' is defined as an activity between two or more participants according to a set of rules, at the end of which each participant either gets some benefit or suffers some loss.

Basic Terms

A. **Player –** A participant is called a Player.

B. **Play –** A Play of the game is said to occur when each Player has chosen a course of action.

C. **2 Person Game –** If the number of Players in a Game is two then it is called 2 Person Game. The term Person refers to an individual or a group aiming at a particular objective.

D. **N Person Game –** If the number of Players in a Game is N (where $N > 2$) then it is called N Person Game.

E. **Zero Sum Game –** If the sum of the amounts won by all winners is equal to that lost by all losers then the game is called Zero Sum Game. In other words, sum of the gains and losses in such a game is zero. When there are only two players participating in a game which has resulted in zero sum then it is called 2 Person Zero Sum Game. For a game with zero sum and N participants, we call it as N Person Zero Sum Game.

   An example of 2 Person Zero Sum Game is the competition of two firms who are trying to increase their share of the market. Here gain of the market share of one will almost be equal to the other's loss of the market share.

Two Person Zero Sum Games are also called rectangular game because their payoff matrix is in the rectangular form.

F. **Non Zero Sum Game –** If the sum of the gains or losses in a game is not equal to zero then it is called a Non Zero Sum Game. An example of such a game is the competition between two firms for increasing their respective market shares through intensive advertising campaigns. Due to the advertisement of their product, both the firms might gain market share which may not be of equal magnitude, but at the same time gain of one is not exactly equal to the loss of the other. Hence, sum of the amounts is not zero.

G. **Strategy:** It is the predetermined rule by which a Player while playing decides the course of action from his own list of courses of action. There are two types of strategies – Pure and Mixed.

H. **Pure Strategy –** If a Player knows exactly what the other Player is going to do, a deterministic situation is obtained. The objective is to maximize the gain. Thus, it is a decision in advance of all plays always to choose a particular course of action. A pure strategy is usually represented by a number with which the course of action is associated.

I. **Mixed Strategy –** If a Player is guessing as to which activity is to be selected by the other on any particular occasion, a probabilistic situation is obtained. The objective in this case is to maximize the expected gain. Thus, it is a decision, in advance of all plays to choose a course of action for each play in accordance with some particular probability distribution. When a player decides in advance to use his available courses of action in some fixed proportion, he is said to use mixed strategy. In other words we can say that the Mixed Strategy is a selection among Pure Strategies with some fixed probabilities.

J. **Payoff –** The outcome of playing the game is known as Payoff. It is the quantitative measure of satisfaction a Player gets at the end of each play. For a business situation, the measure of satisfaction mentioned above could be increase in profits, Expansion in actual market share etc. In other words, it is the net gain a course of action or strategy brings to a player for any counter course of action or strategy of the competitor.

K. **Payoff Matrix –** This is a tabular representation showing the outcomes or payoffs corresponding to different strategies of the participating Players. Since a Game involves at least two Players, the table referred above always forms a matrix with some rows (m, say) and columns (n, say). Rules of a Payoff Matrix are –

- Rows denote the activities or courses of action available to Player A who is considered as the maximising player.
- Columns denote the activities or courses of action available to Player B who is the minimising player.
- Figure shown in the cell x_{ij} denotes payment to A when he chooses the activity i against B's choice of activity j.
- For a Two Person Zero Sum Game, any cell entry of the Player B's payoff matrix will be negative of the corresponding cell entry in the Player A's payoff matrix, so that the sum of the payoffs of the two Players is ultimately zero.

The following table is an example of a Payoff Matrix of a Two Person Zero Sum Game which says that two firms are competing for business with the mentioned strategies so that one's gain is another's loss:

Strategies of Firm A	Strategies of Firm B		
	No advertising	Medium advertising	Heavy advertising
No advertising	10	5	-2
Heavy advertising	16	14	10

Here a positive payoff denotes gain to the maximising player i.e. Firm A (shown as Row) and loss to the minimising player i.e. Firm B (shown as Column). If Firm A chooses strategy "No advertising" and Firm B chooses strategy "Medium advertising" then gain of A will be 5 and loss of B will also be 5.

L. **Optimal Strategy:** A course of action which puts the Player in the most preferred position, irrespective of the strategy of his competitors, is called Optimal Strategy. Not opting for this strategy will result in decreased payoff of the player.

M. **Value of the Game:** It is the expected payoff of play when all the players of the game follow their optimal strategies. The game is called Fair if the value of the game is zero and Unfair if it is non-zero.

Assumptions in a Game

1. The players act rationally and intelligently.
2. Each player has a finite set of strategies available to him.
3. The players attempt to maximise gains and minimise losses.
4. All relevant information is available to each player.
5. The players take individual decisions without direct communication with each other.
6. The players select their strategies simultaneously.
7. The payoff is fixed and determined in advance.

Solution of Pure Strategy Games with Saddle Point

Pure Strategy Games are solved using Maximin – Minimax criteria. The maximising player (whose strategies are shown along the rows of the Payoff Matrix) arrives at his optimal strategy on the basis of Maximin criteria and the minimising player (whose strategies are shown along the columns of the Payoff Matrix) follows Minimax criteria. The game is solved when Maximin and Minimax values are equal.

Maximin value is determined as follows –

(i) Find minimum value in each row of the given payoff matrix. This denotes minimum possible gain against each strategy of the Maximising Player.

(ii) Maximin value is the maximum of these minimum values.

Minimax value is determined as follows –

(i) Find maximum value in each column of the given payoff matrix. This denotes maximum possible loss against each strategy of the Minimising Player.

(ii) Minimax value is the minimum of these maximum values.

Saddle Point is said to exist when the Maximin and Minimax values are equal. Thus, Saddle Point is the position of such an element in the payoff matrix, which is minimum in its row and maximum in its column. The Saddle Point is the solution or Value of the game. The strategies of the two players corresponding to the Saddle Point are their optimal strategies. If there is more than one Saddle Point then more than one solution will be possible corresponding to each Saddle Point.

Following Payoff Matrix is used to illustrate 2 Person Zero Sum Pure Strategy Game:

		Strategies of Player B		Row Minimum
		B_1	B_2	
Strategies of Player A	A_1	9	2	2
	A_2	8	6	**6 = Maximin**
	A_3	6	4	4
Column Maximum		9	**6 = Minimax**	

In the table above, A is the maximising player with strategies represented along the rows and B is the minimising player with strategies represented along the columns.

Suppose the player A starts the game knowing fully well that whatever strategy he adopts B will select that particular counter strategy which will minimise the payoff to A. If A selects A_1 then B will definitely select B_2 so that A gets minimum possible gain i.e. 2 under the situation. Similarly if A chooses A_2 then B will go for B_2 and so on. Thus, A wants to maximise his gain which is possible by going for the maximum value among the Row minimums or the Maximin value. Similarly, B wants to minimise his loss which is the minimum among the Column maximums or the Minimax value.

We observe here that both Maximin and Minimax values are equal to 6. Hence there exists a Saddle Point. Also this value corresponds to the cell A_2B_2. That means the Optimal strategy for the Player A is A_2 and that for the Player B is B_2. Value of the Game is 6 for A and -6 for B which means the game is Zero Sum.

Principle of Dominance

According to the Principle of Dominance if any strategy of a player dominates over his another strategy in all conditions then the later can be ignored because it will not affect the solution of the game. A strategy dominates over the other only if it is preferable over the other in all conditions. From the gainer's point of view, if a strategy gives more gain than another strategy for all strategies of the loser, then the first strategy dominates over the other and the second one can be ignored altogether. Similarly from the loser's point of view, if a strategy involves lesser loss than the other in all conditions, the second one can be omitted without affecting decision. So determination of superior or inferior strategy depends upon the objective of the player. Since each player has to select his best strategy, the inferior strategies can be eliminated. In other words, ineffective rows and columns can be deleted from the given payoff matrix so that its size is reduced.

For deleting the ineffective rows and columns, the following Rules are used –

Rule 1 – If all the elements of a row (say ith row) of a payoff matrix are less than or equal to the corresponding elements of another row (say jth row) then the Maximising Player will never choose the ith strategy. In other words ith strategy is dominated by the jth strategy.

Rule 2 – If all the elements of a column (say pth column) of a payoff matrix are more than or equal to the corresponding elements of another column (say qth column) then the Minimising Player will never choose the pth strategy. In other words, pth strategy is dominated by the qth strategy.

Rule 3 – A pure strategy may be dominated if it is inferior to average of two or more other pure strategies. If all the elements of a row are less than or equal to the average of the corresponding elements of two or more other rows then this row is said to be dominated by the other group of rows for which average is computed. Similar concept is also applicable for column with the exception of having its elements more than the average of the corresponding elements of two or more columns.

Principle of Dominance can be applied to both Pure Strategy as well as Mixed Strategy problems. Its basic objective is to reduce the size of the given Payoff Matrix. Aim should always be made to get a (2 × 2) matrix by using this Principle.

Solution of Mixed Strategy Games

Any problem of Game without a Saddle Point is considered to be the problem of Mixed Strategy. In such cases both players will use various strategies with certain probabilities to optimize. Unlike Pure Strategy problems (where a single strategy will certainly be the optimum one) here we need to find out the probabilities of various strategies of both the players as well as expected value of the game. Games with Mixed Strategy are solved by the following methods depending on the size of the Payoff Matrix.

- (2 × 2) Game – Odds Method or Arithmetic Method
- Dominance Method (applicable for m × n Payoff Matrix convertible to 2 × 2 Payoff Matrix by application of Rules of Dominance)
- (2 × n) and (m × 2) Game – Graphical Method

1. Odds Method

Odds Method is applicable if and only if the Payoff Matrix is of size (2 × 2). Odds are nothing but the magnitude (i.e. without sign or ignoring negative sign, if any) of the differences of the elements of various rows as well as columns. Method of calculating Odds is given below –

1. Find out magnitude of difference in the values of cell (1,1) and (1,2) of the 1st Row and place it against the 2nd Row.
2. Compute magnitude of the difference in the cell entries of (2,1) and (2,2) of the 2nd Row and put it against the 1st Row.
3. Compute magnitude of the difference in the cell entries of (1,1) and (2,1) of the 1st Column and put it below the 2nd Column.
4. Compute magnitude of the difference in the cell entries of (1,2) and (2,2) of the 2nd Column and put it below the 1st Column.
5. Ensure that the sum of the differences calculated for the Rows is equal to that for the columns. In other words Sum of the differences calculated in steps (1) and (2) should be equal to that calculated in steps (3) and (4).

Note – Only the magnitude of the differences should be taken into account ignoring the negative signs, if any.

		Strategies of Y		
		Y_1	Y_2	ODDs
Strategies of X	X_1	a_1	a_2	$(b_1 - b_2)$
	X_2	b_1	b_2	$(a_1 - a_2)$
	ODDs	$(a_2 - b_2)$	$(a_1 - b_1)$	

Probabilities of X as well as Y taking different strategies are calculated by using the following formulae –

$P(X_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)]$ and $P(X_2) = (a_1 - a_2) \div [(b_1 - b_2) + (a_1 - a_2)]$

$P(Y_1) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)]$ and $P(Y_2) = (a_1 - b_1) \div [(a_2 - b_2) + (a_1 - b_1)]$

Value of the Game is determined using the formula:- $v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)]$

[Note: $P(X_1) + P(X_2) = 1$ and $P(Y_1) + P(Y_2) = 1$. So once $P(X_1)$ is calculated, $P(X_2)$ can always be calculated as complement of $P(X_1)$ instead of going for the formula. Similar is the case for $P(Y_2)$.]

2. Dominance Method

Dominance Method is applied for reducing the size of (m × n) Payoff Matrix (when either one of m and n or both m and n are greater than 2) when there exist no Saddle Point. The aim is to get (2 × 2) Matrix, so that Odds Method can be applied to find the Probabilities and the Value of the Game as described above. It can be mentioned that the strategies which are dominated by the others and ultimately ignored will not be used by the players and hence their probabilities will be zero.

Illustration 1

Solve the Game with the Payoff Matrix $\begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$

Solution:

Let the given Game is played by the Players A and B with A (the maximising player) having strategies A_1 and A_2 represented along the rows and B (the minimising player) having strategies B_1 and B_2 represented along the columns. So the given Payoff Matrix can be written as follows –

		Strategies of B		
		B_1	B_2	Row Min.
Strategies of A	A_1	1	5	1
	A_2	4	2	2 = Maximin
	Column Max.	4 = Minimax	5	

Maximin value (2) ≠ Minimax value (4). Thus, Saddle Point does not exist. So this is a problem of Mixed Strategy with (2 × 2) Payoff Matrix.

We apply Odds Method to solve the problem. Odds are calculated as follows –

		Strategies of B		
		B_1	B_2	ODDs
Strategies of A	A_1	$1 = a_1$	$5 = a_2$	$b_1 - b_2 = 4-2 = 2$
	A_2	$4 = b_1$	$2 = b_2$	$a_1 - a_2 = 1-5 = 4$
	ODDs	$a_2 - b_2 = 5- 2 = 3$	$a_1 - b_1 = 1-4 = 3$	

[**Note**: Though $(a_1 - b_1) = 1 - 4 = -3$, but here it has been taken +3 as per the concept of Odds, similar is the case for $(a_1 - a_2)$]

Probabilities of A and B taking their different strategies are calculated as follows –

$P(A_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 2 / [2 + 4] = 2/6 = 1/3$

$P(A_2) = (a_1 - a_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 4 / [2 + 4] = 4/6 = 2/3$

$P(B_1) = (a_2 – b_2) \div [(a_2 – b_2) + (a_1 – b_1)] = 3 / [3 + 3] = 3/6 = 1/2$

$P(B_2) = (a_1 – b_1) \div [(a_2 – b_2) + (a_1 – b_1)] = 3 / [3 + 3] = 3/6 = 1/2$

Value of the Game = $v = [a_1(b_1 – b_2) + b_1(a_1 – a_2)] \div [(b_1 – b_2) + (a_1 – a_2)] = [1\times2 + 4\times4] \div [2 + 4] = 18/6 = 3$

So A chooses his strategies (A_1, A_2) with probabilities (1/3, 2/3) & B chooses his strategies (B_1, B_2) with probabilities (1/2, 1/2) and Value of the Game = 3

[**Note:** Calculated Value of the Game is the Expected Gain of A which is same as the Expected Loss of B]

Illustration 2

The Management of a company is negotiating with its Union for revision of hourly wages of its employees. The Management deployed a Consulting Firm who has prepared a payoff matrix for the purpose which indicates the additional hourly cost (in ₹) to the company. It is shown below: you being a part of the Consulting Firm have to assist the Management in selecting the best strategy. What is the value of the game? How is it going to affect the company's cost?

Management's Strategies	Strategies of the Union			
	U_1	U_2	U_3	U_4
M_1	2.50	2.70	3.50	– 0.20
M_2	2.00	1.60	0.80	0.80
M_3	1.40	1.20	1.50	1.30
M_4	3.00	1.40	1.90	0

Solution:

As the Management's objective is to minimise the cost, they can be considered as the Minimising Player and the Union as the Maximising Player in this problem of Game. Thus, to solve the problem we have to recast the given Payoff Matrix by transposing it as below:-

Strategies of the Union	Management's Strategies				Row Minimum
	M_1	M_2	M_3	M_4	
U_1	2.50	2.00	1.40	3.00	1.40 = Maximin
U_2	2.70	1.60	1.20	1.40	1.20
U_3	3.50	0.80	1.50	1.90	0.80
U_4	-0.20	0.80	1.30	0	-0.20
Column Maximum	3.50	2.00	1.50 = Minimax	3.00	

Maximin value (1.40) ≠ Minimax value (1.50). Thus, Saddle Point does not exist. So this is a problem of Mixed Strategy. Since the matrix is not a (2×2) Matrix, Dominance Rules are applied to reduce its size to make it a (2×2) Matrix.

As all the elements of the 3rd Row of the above matrix are either greater than or equal to the corresponding elements of the 4th Row, the 3rd Row can be considered to dominate the 4th. So the 4th Row is ignored and the new matrix is shown below.

Strategies of the Union	Management's Strategies			
	M_1	M_2	M_3	M_4
U_1	2.50	2.00	1.40	3.00
U_2	2.70	1.60	1.20	1.40
U_3	3.50	0.80	1.50	1.90

Again all the elements of the 1st Column are greater than the corresponding elements of the 2nd Column, the 1st Column is dominated by the 2nd Column. Hence the 1st Column is ignored and the new matrix is shown below.

Strategies of the Union	Management's Strategies		
	M_2	M_3	M_4
U_1	2.00	1.40	3.00
U_2	1.60	1.20	1.40
U_3	0.80	1.50	1.90

All the elements of the 3rd Column (i.e. for Strategy M_4) of this matrix are more than the corresponding elements of 2nd Column (i.e. for Strategy M_3). Hence M_4 is dominated by M_3 and ignored. The new matrix is shown below.

Strategies of the Union	Management's Strategies	
	M_2	M_3
U_1	2.00	1.40
U_2	1.60	1.20
U_3	0.80	1.50

Again all the elements of the 1st Row (for strategy U_1) are greater than the corresponding elements of the 2nd Row (for strategy U_2). So U_2 is dominated by U_1 and ignored. The new matrix is shown below.

Strategies of the Union	Management's Strategies	
	M_2	M_3
U_1	2.00	1.40
U_3	0.80	1.50

This is a (2 × 2) Matrix. Now the problem of Game is solved by using Odds Method. Odds are calculated as below.

Strategies of the Union	Management's Strategies		ODDs
	M_2	M_3	
U_1	$2.00 = a_1$	$1.40 = a_2$	$b_1 - b_2 = 0.80 - 1.50 = 0.70$
U_3	$0.80 = b_1$	$1.50 = b_2$	$a_1 - a_2 = 2.00 - 1.40 = 0.60$
ODDs	$a_2 - b_2 = 1.40 - 1.50 = 0.10$	$a_1 - b_1 = 2.00 - 0.80 = 1.20$	Sum of the ODDs = 1.30

Probabilities of the Union and the Management taking their different strategies are calculated as follows –

$P(U_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 0.70 / [0.70 + 0.60] = 0.70/1.30 = 7/13$

$P(U_3) = (a_1 - a_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 0.60 / [0.70 + 0.60] = 0.60/1.30 = 6/13$

$P(M_2) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)] = 0.10 / [0.10 + 1.20] = 0.10/1.30 = 1/13$

$P(M_3) = (a_1 - b_1) \div [(a_2 - b_2) + (a_1 - b_1)] = 1.20 / [0.10 + 1.20] = 1.20/1.30 = 12/13$

Value of the Game = $v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)] = [2.00\times0.70 + 0.80\times0.60] \div [0.70 + 0.60]$

$= [1.40 + 0.48] / 1.30 = 1.88/1.30 = 1.45$

So the Union chooses its Strategies U_1, U_2, U_3 & U_4 with probabilities (7/13, 0, 6/13, 0) and the Management chooses its Strategies M_1, M_2, M_3 & M_4 with probabilities (0, 1/13, 12/13, 0).

Expected Gain to the Union is ₹ 1.45 and the corresponding Loss to the Management is ₹ 1.45.

Thus, the hourly cost of the company will increase by ₹ 1.45

Illustration 3

Solve the Game using Dominance Principle $\begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix}$

Solution:

Let the given Game is played by the Players A and B with A (the maximising player) having strategies A_1, A_2 and A_3 represented along the rows and B (the minimising player) having strategies B_1, B_2 and B_3 represented along the columns. So the given Payoff Matrix can be written as follows –

	Strategies of B		
Strategies of A	B_1	B_2	B_3
A_1	15	2	3
A_2	6	5	7
A_3	-7	4	0

All the elements of Row A_3 are less than the corresponding elements of Row A_2. So A_3 is dominated by A_2. Hence it is ignored and deleted. The new matrix is given below.

Strategies of A	Strategies of B		
	B_1	B_2	B_3
A_1	15	2	3
A_2	6	5	7

Here all the elements of B_3 are more than the corresponding elements of B_2. Hence B_3 is dominated by B_2 and ignored to get the new matrix below.

	Strategies of B		
Strategies of A	B_1	B_2	Row Min.
A_1	15	2	2
A_2	6	5	**5**
Column Max.	15	**5**	

Maximum among the Row minimums = 5 = Maximin value and Minimum among the Column maximums = 5 = Minimax value. As, Maximin and Minimax values are equal, there exists a Saddle Point. It occurs at the cell A_2B_2.

Hence optimal strategies of A and B are respectively A_2 and B_2. Also value of the Game = 5

[NOTE – This is a problem of Pure Strategy and could have been solved without the use of Dominance Rules, but the question has specifically asked for the usage of Dominance Rules. So the same is used.]

Illustration 4

Joy Givers and Milan Toys are the two toy manufacturers who always compete with each other to increase their respective market shares. For both the companies the Marketing team work with close coordination with the Design team and always come out with attractive toys which are normally in great demand. To meet the demand, they have various strategic options like working for 8 hours a day, 12 hours a day, 16 hours a day, 24 hours a day, Subcontracting etc. which will ultimately increase the market share. Joy Givers have decided not to go for all the above mentioned options and set up the following payoff matrix in which the percentage increase in market share is given against different strategies of Milan Toys

STRATEGIES of	Milan Toys			
Joy Givers	Working 8 hrs/day	Working 12 hrs/day	Working 16 hrs/day	Subcontracting
Working 12 hrs/day	8	10	9	14
Working 16 hrs/day	10	11	8	12
Working 24 hrs/day	13	12	14	13

Use Principle of Dominance to find the Optimal Strategies of the two manufacturers and the value of the Game.

Solution:

Joy Givers is the Maximising player with strategies represented along the rows and Milan Toys is the Minimising Player with strategies represented along the columns. For ease of representation we consider the respective strategies of Joy Givers as J_1, J_2 & J_3 and those of Milan Toys as M_1, M_2 & M_3.

STRATEGIES of	Milan Toys			
Joy Givers	M_1	M_2	M_3	M_4
J_1	8	10	9	14
J_2	10	11	8	12
J_3	13	12	14	13

All the elements of 4th Column are either greater than or equal to the corresponding elements of the 1st Column. So 4th Column's strategy (M_4) is dominated by the 1st Column's strategy (M_1). Hence M_4 is ignored. The new matrix is given below.

STRATEGIES	Milan Toys		
Joy Givers	M_1	M_2	M_3
J_1	8	10	9
J_2	10	11	8
J_3	13	12	14

All the elements of 1st Row are less than the corresponding elements of the 3rd Row. Thus, strategy of 1st Row i.e. J_1 is dominated by the strategy of the 3rd Row i.e. J_3 and ignored. The reduced matrix becomes –

STRATEGIES	Milan Toys		
Joy Givers	M_1	M_2	M_3
J_2	10	11	8
J_3	13	12	14

Apparently first two rules of dominance cannot be applied to either of the rows or columns of the above matrix, but if the average of the elements of the strategies M_2 and M_3 be taken then we get a matrix shown below.

STRATEGIES	Milan Toys	
Joy Givers	M_1	$[M_2 + M_3]/2$
J_2	10	$(11 + 8)/2 = 9.5$
J_3	13	$(12 + 14)/2 = 13$

So the elements of the strategy M_1 are either more or equal to the average of the corresponding elements of M_2 and M_3. Hence M_1 is dominated by M_2 and M_3. Thus, M_1 is deleted and the reduced matrix is as below.

STRATEGIES of	Milan Toys		Row Minimum
Joy Givers	M_2	M_3	
J_2	11	8	8
J_3	**12***	14	12 = Maximin
Column Maximum	12 = Minimax	14	

So Maximin value = 12 = Minimax value. Hence there exists a Saddle Point at the junction J_3M_2

Thus, optimal strategy of Joy Giver is J_3 that is "Working 24 hours /day" and that for Milan Toys is M_2 that is "Working 12 hours/day". Value of the Game = 12 (which means a 12% increase in market share for Joy Givers)

3. Graphical Method

Graphical Method is applied to solve (2 × n) and (m × 2) Game problems, when both m and n are more than 2. Since the optimal strategies for both the players assign non zero probabilities to the same number of pure strategies, it is obvious that if one player has only two strategies the other will also use two strategies. Graphical method facilitates to find out which of the two strategies can be used. When Rules of Dominance cannot be applied to a payoff matrix of size (2 × n) or (m × 2) then Graphical Method is used.

Following are the steps for solving a (2 × n) Game –

1. Draw two vertical lines 1 unit apart along a horizontal line to represent the axes $x_1= 0$ and $x_1= 1$ & mark a suitable scale on each one.
2. Take the values in the first Row of the Payoff Matrix and plot each one as a point on the scale of the vertical line $x_1= 1$.
3. Take the values in the second Row of the Payoff Matrix & plot each one as a point on the scale of the vertical line $x_1= 0$.
4. The point a_{1j} on the line $x_1 = 1$ should be joined to the point a_{2j} on the line $x_1 = 0$ to get a straight line.
5. Draw n such straight lines for j = 1, 2, 3,.......... n. Each of these lines represents the expected payoff of the maximising player (whose 2 strategies are represented by the rows) against n different strategies of the minimising player (whose strategies are represented by the columns).
6. Mark the lower envelope of the area obtained by drawing these n straight lines.
7. The highest point of the lower envelope is the Maximin point.
8. The straight lines passing through this Maximin point corresponds to the optimum strategies of the minimising player. All the other strategies of the minimising player should be ignored.
9. So now the desired (2 × 2) payoff matrix is obtained, the Game can be solved using the method of Odds.

The steps for solving (m × 2) Game are almost similar to those of (2×n) Game. The main difference with respect to the previous case lies in the fact that we have to consider the values in the First and the second Columns and plot them as points on the two axes $x_1 = 0$ and $x_1 = 1$ so that m straight lines can be drawn representing expected payoff of the minimising player against the strategies of the maximising player. Now the upper envelope of the common area bounded by the straight lines should be marked. Lowest point of this envelope represents Minimax point. The strategy lines which intersect to give the Minimax point are the optimum strategies of the maximising player. All the other strategies of the maximising player should be ignored to give a (2×2) payoff matrix which can be solved by using the method of Odds.

[**Note**: Problems of Game, with Payoff Matrix of size (m×n) when both m & n > 2, are solved using Simplex Method of Linear Programming]

Illustration 5

Solve the Game represented by the payoff matrix :- $\begin{bmatrix} 1 & 3 & 12 \\ 8 & 6 & 2 \end{bmatrix}$

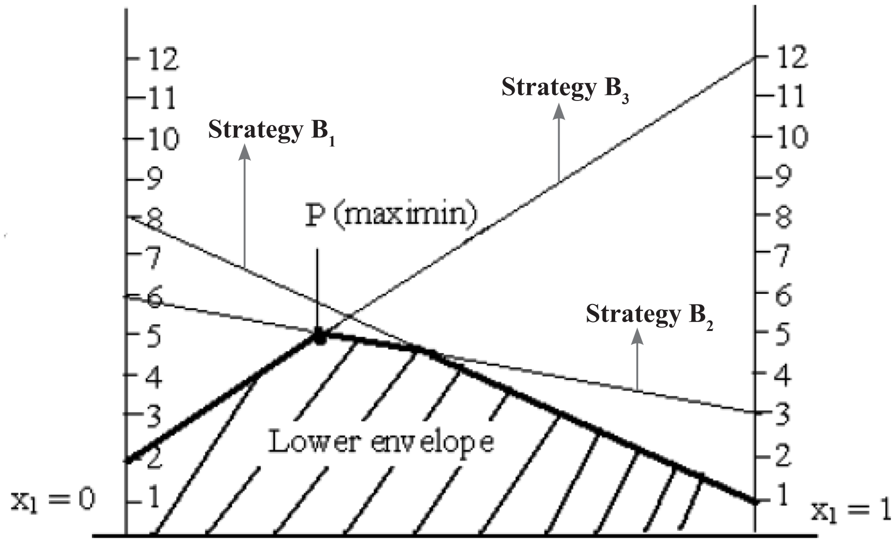
Solution:

Let the given Game is played by the Players A and B with A (the maximising player) having strategies A_1 and A_2 represented along the rows and B (the minimising player) having strategies B_1, B_2 and B_3 represented along the columns. So the given Payoff Matrix can be written as follows –

	Strategies of B			Row Minimum
Strategies of A	B_1	B_2	B_3	
A_1	1	3	12	1
A_2	8	6	2	2 = Maximin
Column Maximum	8	6 = Minimax	12	

Maximin value (2) ≠ Minimax value (6). Thus, Saddle Point does not exist. So this is a problem of Mixed Strategy. Since the matrix is not a (2×2) Matrix, we check the possibility of applying Dominance Rules to reduce its size to (2×2) Matrix, but it is observed that the Dominance Rules are also not suitable for reducing the size of the given matrix. Hence we go for solving the problem using Graphical Method suitable for (2×n) matrix because the given matrix is (2×3)

As shown below two vertical lines are drawn on a horizontal line 1 unit apart to represent the axes $x_1 = 0$ & $x_1 = 1$ and marked them to a scale. Now the values 1, 3 and 12 of the 1st Row of the given matrix are plotted as points on the axis $x_1 = 1$ and the values 8, 6 and 2 of the 2nd Row are plotted as points on the axis $x_1 = 0$. Then the pair of points 12 & 2, 3 & 6 and 1 & 8 are joined with the help of straight lines. These lines represent the expected payoff of Player A against the strategies B_3, B_2 & B_1 respectively of Player B. The lower envelope of the area bounded by these lines and the axes is shaded as shown. Highest point P of this envelope is the Maximin point. As P is the point of intersection of A's expected payoff lines against strategies B_2 & B_3 of B, we can say that B will opt for these two strategies and ignore strategy B_1.



So the reduced payoff matrix is given as follows –

Strategies of A	Strategies of B		
	B_2	B_3	ODDs
A_1	$3 = a_1$	$12 = a_2$	$b_1 - b_2 = 6 - 2 = 4$
A_2	$6 = b_1$	$2 = b_2$	$a_1 - a_2 = 3 - 12 = 9$
ODDs	$a_2 - b_2 = 12 - 2 = 10$	$a_1 - b_1 = 3 - 6 = 3$	Sum of the ODDs = 13

Probabilities of A and B taking their different strategies are calculated as follows –

$P(A_1) = (b_1 - b_2) \div [(b_1 - b_2) + (a_1 - a_2)] = 4 / [4 + 9] = 4/13$, $P(A_2) = 1 - P(A_1) = 1 - 4/13 = 9/13$

$P(B_2) = (a_2 - b_2) \div [(a_2 - b_2) + (a_1 - b_1)] = 10 / [10 + 3] = 10/13$, $P(B_3) = 1 - P(B_2) = 1 - 10/13 = 3/13$

Value of the Game $= v = [a_1(b_1 - b_2) + b_1(a_1 - a_2)] \div [(b_1 - b_2) + (a_1 - a_2)] = [3\times4 + 6\times9] \div [4 + 9] = 66/13$

So A chooses the strategies with probabilities (4/13, 9/13) and B chooses the strategies with probabilities (0, 10/13, 3/13).

Limitations of Game Theory

1. The assumption that the players have the knowledge about their own and the opponent's payoffs is unrealistic. He can only make a guess of his own and the opponent's strategies.
2. As the number of players increase in the game, the analysis of the strategies becomes increasingly complex and difficult. In practice there are many firms in an oligopoly situation where game theory is not useful.
3. The assumptions of Pure Strategy game show that the players are risk averse and have complete knowledge about each other's strategies. This is impractical.
4. Rather than each player in an oligopoly situation working under uncertain conditions, the players will allow each other to share the secrets of business in order to work out a collusion. So the mixed strategies are also not very useful.

EXERCISE

A. Theoretical Questions:

Multiple Choice Questions

1. Two person zero sum game means that

   (a) The sum of losses of one player is equal to the sum of the gains of the other

   (b) The sum of losses of one player may not be equal to the sum of the gains of the other

   (c) No player gains or loses

   (d) None of the above

2. Game theory models are classified by the

   (a) Number of players

   (b) Sum of all payoffs

   (c) Number of strategies

   (d) All of these

3. A game is said to be unfair if

   (a) Upper and lower values of the game are not equal

   (b) Upper and lower values of the game are equal and the sum is zero

   (c) Option (a) is correct but not Option (b)

   (d) Option (b) is correct but not Option (a)

4. What happens when the maximin and minimax values of the game are equal?

   (a) No solution exists

   (b) Solution is mixed

   (c) Saddle point exists

   (d) None of these

5. A mixed strategy game can be solved by

   (a) Arithmetic method

   (b) Graphical method

   (c) Dominance method

   (d) All of these

6. The size of the payoff matrix of a game can be reduced by using the principle of

   (a) Game inversion

   (b) Rotation reduction

   (c) Dominance

   (d) Game transpose

7. The payoff value for which each player in a game always selects the same strategy is called the
   (a) Saddle point
   (b) Equilibrium point
   (c) Both option (a) and option (b)
   (d) None of the above

8. Games which involve more than two players are called
   (a) Conflicting games
   (b) Negotiable games
   (c) N person game
   (d) All of these

9. When the sum of the gains of one player is equal to the sum of the losses to another player then it is called
   (a) Fair game
   (b) Zero sum game
   (c) Both option (a) and option (b)
   (d) Only option (b) and not option (a)

10. When no saddle point is found in the payoff matrix of a game, the value of the game is found by
   (a) Reducing the size of the game to apply the odds method
   (b) Solving any one of the (2×2) sub game
   (c) Finding the average of the values of the payoff matrix
   (d) None of these

11. A saddle point exists when
   (a) Maximin value = Maximax value
   (b) Minimax value = Minimum value
   (c) Minimax value = Maximin value
   (d) Minimax value = Minimin value

12. In a pure strategy game
   (a) Any strategy can be selected arbitrarily
   (b) A particular strategy is selected by each player
   (c) Both players select their optimal strategy
   (d) None of these

13. In a mixed strategy game
   (a) No saddle point exists

(b) Each player selects the same strategy without considering the choice of the other

(c) Each player always selects the same strategy

(d) None of these

14. Game theory is the study of

(a) Selecting optimal strategies

(b) Resolving conflict between players

(c) Giving equal outcome to the participants

(d) None of the above

15. If the value of the game is zero, then the game is known as

(a) Fair strategy

(b) Pure strategy

(c) Pure game

(d) Mixed strategy

16. The games with saddle points are

(a) Probabilistic in nature

(b) Normative in nature

(c) Stochastic in nature

(d) Deterministic in nature

17. When the game is played on a predetermined course of action, which does not change throughout the game then it is known as

(a) Pure strategy game

(b) Fair strategy game

(c) Mixed strategy game

(d) Unsteady game

18. If the losses of Player A are the gains of Player B, then it is called

(a) Lump sum game

(b) Zero sum game

(c) Unfair game

(d) None of the above

19. Identify the incorrect one

(a) A game without saddle point is probabilistic

(b) Game with saddle point will have pure strategies

(c) Game with saddle point cannot be solved with dominance rule

(d) Game without saddle point has mixed strategies

20. In case there is no saddle point in a game then the game is
    (a) Deterministic game
    (b) Fair game
    (c) Mixed strategy game
    (d) Multi player game

21. When Minimax and Maximin criteria matches then
    (a) A fair game exists
    (b) An unfair game exists
    (c) Mixed strategy exists
    (d) Saddle point exists

22. When there is dominance in a game then
    (a) Least of the row $\geq$ Highest of another row
    (b) Least of the row $\leq$ Highest of another row
    (c) Every element in a row $\geq$ Corresponding element of another column
    (d) Every element in a row $\leq$ Corresponding element of another row

23. A game is played when
    (a) The manager gives signal
    (b) Each player chooses one of his courses of action simultaneously
    (c) The player who comes to the field first says he will start the game
    (d) When the latecomer starts the game

24. In a game the list of the courses of action with each player is
    (a) Finite
    (b) Infinite
    (c) Only 3
    (d) None of the above

25. When the game is having a saddle point then the method used to solve the game is
    (a) Linear Programming method
    (b) Minimax and Maximin criteria
    (c) Odds method
    (d) Graphical method

26. Linear Programming method should be used to determine the value of the game when the size of the payoff matrix is

(a) 2×2

(b) 3×4

(c) m×2

(d) 2×n

27. If there are more than two persons in a game then the game is known as

(a) Non zero sum game

(b) Open game

(c) Multiplayer game

(d) Big game

28. A competitive situation is known as

(a) Competition

(b) Marketing

(c) Game

(d) All the above

29. Which one of the following is an assumption of Game Theory?

(a) All players act rationally and intelligently

(b) The winner alone acts rationally

(c) The loser acts intelligently

(d) Both believes in luck

30. For the Payoff Matrix $\begin{pmatrix} -5 & -2 \\ 10 & 5 \end{pmatrix}$ the maximising player always uses

(a) The first strategy

(b) Average of the two strategies

(c) The second strategy

(d) All the above strategies

Answers:

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
a	**d**	**a**	**c**	**d**	**c**	**a**	**c**	**d**	**a**	**c**	**c**	**a**	**a**	**a**
16	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
d	**a**	**b**	**c**	**c**	**d**	**d**	**b**	**a**	**b**	**b**	**c**	**c**	**a**	**c**

State True or False

1. The name Game is derived from the fact that the end result always gives lot of fun to the players.
2. Strategies are the different courses of action of the players.
3. In Pure Strategy the objective of the maximizing player is to maximize the Gain, but in Mixed Strategy the objective is to maximize the Expected Gain.
4. Both Pure and Mixed Strategy problems can be solved by the Rules of Dominance.
5. A fair game results when the value of the game is zero.
6. Mixed Strategy Games are deterministic in nature.
7. Zero sum games always have two participants only.
8. A pure strategy may be dominated if it is inferior to average of two or more other pure strategies.
9. Columns of a payoff matrix represent the strategies of the maximising player
10. Graphical Method can be used for mixed strategy games having any size of payoff matrix.
11. In a game the players act rationally and intelligently.
12. In a (m×2) mixed strategy game the graphical method is used to find out the maximin point.
13. Equality of minimax and maximin values result in the existence of Saddle Point.
14. Optimal strategy is that course of action which puts a player in most preferred position.
15. Military operations are examples of game.

Answers:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F	T	T	T	T	F	F	T	F	F	T	F	T	T	T

Fill in the blanks

1. For the Payoff Matrix $\begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$ the ___ strategy of the minimising player is dominated by the third.
2. "Teesta" water distribution conflict between India and Bangladesh can be considered as a situation of _____.
3. Arithmetic Method is used to solve ______ strategy problems of Game.
4. Value of the Game $\begin{bmatrix} 2 & 3 \\ -5 & 5 \end{bmatrix}$ is ___.
5. As per the Rules of Dominance, a strategy of the maximising player is said to dominate another if all the elements of a row of the payoff matrix are ____ than or equal to the corresponding elements of the other.
6. Pure strategy games are ____ in nature.
7. Dominance principle has ____ rules.
8. Strategies of maximising player of a game are represented along the ____ of the payoff matrix.
9. For (2×n) matrix Mixed strategy Games ____ method is used to find the solution.
10. Columns of the payoff matrix of a game represent the strategies of the ____ player.

11. Game with payoff matrix (m×n) is solved using Simplex Method, if both m and n are greater than ___.
12. In a game, all relevant information is available to ____ player.
13. In (2×2) mixed strategy game, sum of the probabilities of the minimising player taking its two strategies is ___.
14. Pure strategy games always have a ____ point.
15. In problems of game negative payoff of the maximising player indicates a ___.
16. Odds Method is applicable to ____ matrix games only.
17. In practical business situations most of games have ___ than two players.
18. Multiplayer games are also known as ___ person game .
19. In graphical method for (2 × m) game the Maximin value is determined from the ____ point of the lower envelope.
20. A participant of any game is called a _____.

Answers:

1.	**Second**	**2.**	**Game**
3.	**Mixed**	**4.**	**2**
5.	**Greater**	**6.**	**Deterministic**
7.	**Three**	**8.**	**Row**
9.	**Graphical**	**10.**	**Minimising**
11.	**2**	**12.**	**Each**
13.	**1**	**14.**	**Saddle**
15.	**Loss**	**16.**	**(2×2)**
17.	**More**	**18.**	**N**
19.	**Highest**	**20.**	**Player**

B. Numerical Questions

Comprehensive Numerical Problems

1. Find the optimal strategies of the Players for the game having payoff matrix. What is the value of the Game?

Strategies	B_1	B_2	B_3	B_4
A_1	1	7	3	4
A_2	5	6	4	5
A_3	7	2	0	3

2. Solve the game $\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}$

3. Reduce the following game by Dominance Rules and Solve it.

		Strategies of Minimising Player				
		P	Q	R	S	T
Strategies of Maximising Player	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

4. Solve the game $\begin{pmatrix} 3 & -4 \\ 2 & 5 \\ -2 & 8 \end{pmatrix}$ using Graphical method

5. In the suburban area of a large city there are two stores Laxmi Bhandar and Goswami Stores who handle sundry goods. The total number of customers is equally divided between the two due to the fact that the price, quality of goods, services etc. of the two are at par. Assume that a gain of customers for Laxmi Bhander is a loss to Goswami Stores and vice versa. Both the stores plan to run annual sales during the festival period of the year. Sales are advertised through Social Media, Cable TV local channel and Printed Leaflets. Based on the past experience, Laxmi Bhander has prepared the following payoff matrix for the gain or loss in percentage of customers for its different strategies against various counter strategies of Goswami Stores.

	Strategies of Goswami Stores		
Strategies of Laxmi Bhander	Printed leaflets	Cable TV	Social media
Printed leaflets	30	40	– 80
Cable TV	0	15	– 20
Social media	90	20	50

Determine the optimal strategies and worth of such strategies for the stores. What is meant by the cell entry – 80 in the above payoff matrix?

6. Two competing firms (A and B) produce consumer goods of different kind. Among the products one is considered as their bread and butter in terms of the revenue generated. Both the firms are very cautious about the market share for this particular product and keep on doing advertisement campaigns throughout the year to retain the existing customers and also to attract the new ones. For this the marketing teams of both work round the clock and that of A developed data corresponding to varying degrees of advertisement. Same is given below:

   (a) If both the firms take same strategy to counter each other then their market share will be equal.

   (b) Against firm A's strategy of "No marketing" if B goes for "Medium marketing" then A's share of the market will be 40%. For the same strategy of A the market share will be 28% if B takes the strategy "Large marketing"

   (c) Against firm A's strategy of "Medium marketing" if B goes for "No marketing" then A's share of the market will be 70%. For the same strategy of A the market share will be 45% if B takes the strategy "Large marketing"

(d) Against firm A's strategy of "Large marketing" if B goes for "No marketing" then A's share of the market will be 75%. For the same strategy of A the market share will be 47.5% if B takes the strategy "Medium marketing"

Based on the above information prepare the Payoff Matrix. Solve the game problem to get the optimal strategies of the player A. What is the value of the game?

7. Using the data of the above problem prepare the Payoff Matrix for A when you are supplied with the following information.

(a) Selling price of the product = ₹ 4 per unit

(b) Variable cost of the product = ₹ 2.50 per unit

(c) Annual cost for Medium advertising = ₹ 5000

(d) Annual cost for Large advertising = ₹ 15000

(e) Annual sales volume of the product for Firm A = 30000 units

What advertising policy should firm A pursue?

Hints-

Find out the Annual sales volume, for different combination of strategies of A and B. As an example, Annual Sales volume corresponding to A's strategy of "Medium advertisement" and B's strategy of "Large advertisement" is 45% of 30000 = 13500 units

Calculate Annual Profit to the Firm A using the formula below for various combination of strategies of A and B.

Annual Profit = (Selling price – Variable cost) × Annual Sales volume – Annual cost of advertising

Example of this calculation is:-

For A's strategy of "Medium advertisement" and B's strategy of "Large advertisement" the Annual Profit of Firm A is (4 – 2.5) × 13500 – 5000 = ₹ 15250/-

When the Profit figures for all the combinations of strategies of A and B are calculated then the following payoff matrix is obtained.

	Strategies of B		
Strategies of A	**No advertising**	**Medium advertising**	**Large advertising**
No advertising	22500	18000	12600
Medium advertising	26500	17500	15250
Large advertising	18750	6375	7500

From the above matrix we find, against the various strategies of A, the minimum profit figures are as follows –

For No advertising – ₹ 12600

For Medium advertising – ₹ 15250

For Large advertising – ₹ 6375

Thus, to maximise the minimum profit, A should opt for Medium advertising and spend ₹ 5000 per annum

Answers:

1. Optimal strategies A_2 and B_3. Value of the game = 4
2. Strategies of the Maximising Player = (1/5, 4/5) & for the Minimising Player = (3/5, 2/5). Value of the game = 17/5
3. Optimal strategy for the Maximising Player is III and that for the Minimising Player is Q. Value of the game = 5
4. Optimal strategy of the Maximising Player (0.3, 0.7, 0) and for the Minimising Player (0.9, 0.1). Value = 2.3
5. Optimal strategies of Laxmi Bhander = (1/5, 0, 4/5) and for Goswami Stores = (0, 13/15, 2/15), Value of the game = 24

   Cell entry (– 80) means when Laxmi Bhander will take the strategy of distributing Printed Leaflets against the counter strategy of Goswami Stores of Social Media advertisement then they will lose 80% of their customer which will be gained by Goswami Stores.
6. The payoff matrix Showing A's market share is –

	Strategies of B		
Strategies of A	**No advertising**	**Medium advertising**	**Large advertising**
No advertising	50	40	28
Medium advertising	70	50	45
Large advertising	75	47.5	50

Probabilities of A's strategies are (0, 1/3, 2/3). Value of the game = 145/3 = 48.3

Thus, A can expect to have 48.3% market share.

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